STUDYING UNIFORM THICKNESS II: TRANSVERSALLY NON-SIMPLE ITERATED TORUS KNOTS

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Abstract. We prove that an iterated torus knot type fails the uniform thickness property (UTP) if and only if all of its iterations are positive cablings, which is precisely when an iterated torus knot type supports the standard contact structure. We also show that all iterated torus knots that fail the UTP support cabling knot types that are transversally non-simple.

1. Introduction

In this paper, we continue our general study of the *uniform thickness property* (UTP) in the context of iterated torus knots that are embedded in S^3 with the standard tight contact structure. As stated in a previous paper, *Studying uniform thickness I* [L], our goal in this study is to determine the extent to which iterated torus knot types fail to satisfy the UTP, and the extent to which this failure leads to cablings that are Legendrian or transversally non-simple. Motivation for this study is due to the work of Etnyre and Honda [EH1], who showed that the failure of the UTP is a necessary condition for transversal non-simplicity in the class of iterated torus knots. They also established that the $(2,3)$ torus knot fails the UTP and supports a transversally non-simple cabling. In [L] we extended this study of the UTP by establishing new necessary conditions for both the failure of the UTP and transversal non-simplicity in the class of iterated torus knots; in so doing we obtained new families of Legendrian simple iterated torus knots.

The specific goal of this note is to fully answer the first motivating question of our study by providing a complete UTP classification of iterated torus knots, that is, determining which iterated torus knot types satisfy the UTP, and which fail the UTP. We will also address the second motivating question of our study by proving that failure of the UTP for an iterated torus knot type is a sufficient condition for the existence of transversally non-simple cablings of that knot. Specifically, we have the following two theorems and corollary:

Theorem 1.1. *Let $K_r = ((P_1, q_1), ..., (P_i, q_i), ..., (P_r, q_r))$ be an iterated torus knot, where the P_i's are measured in the standard preferred framing, and $q_i > 1$ for all i. Then K_r fails the UTP if and only if $P_i > 0$ for all i, where $1 \leq i \leq r$.*

In the second theorem, $\chi(K)$ is the Euler characteristic of a minimal genus Seifert surface for a knot K:

Theorem 1.2. *If K_r is an iterated torus knot that fails the UTP, then it supports infinitely many transversally non-simple cablings K_{r+1} of the form $(-\chi(K_r), k+1)$, where k ranges over an infinite subset of positive integers.*

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To state our corollary to Theorem 1.1, recall that if K is a fibered knot, then there is an associated open book decomposition of S^3 that supports a contact structure, denoted ξ_K (see [TW]). Iterated torus knots are fibered knots, and Hedden has shown that the subclass of iterated torus knots where each iteration is a positive cabling, i.e. $P_i > 0$ for all i, is precisely the subclass of iterated torus knots where ξ_{K_r} is isotopic to ξ_{std} [He1]. We thus obtain the following corollary:

Corollary 1.3. *An iterated torus knot K_r fails the UTP if and only if $\xi_{K_r} \cong \xi_{std}$.*

We make a few remarks about these theorems. First, it will be shown that these transversally non-simple cablings will all have two Legendrian isotopy classes at the same rotation number and maximal Thurston-Bennequin number $\overline{tb}$, and thus they will exhibit Legendrian non-simplicity at $\overline{tb}$. Second, in the class of iterated torus knots there are certainly more transversally non-simple cablings than those in Theorem 1.2, as evidenced by Etnyre and Honda's example of the transversally non-simple $(2,3)$-cabling of a $(2,3)$-torus knot. However, we present just the class of transversally non-simple cablings in Theorem 1.2, and leave a more complete Legendrian and transversal classification of iterated torus knots as an open question.

We now present a conjectural generalization of the above two theorems and corollary. To this end, recall that Hedden has shown that for general fibered knots K in S^3, $\xi_K \cong \xi_{std}$ precisely when K is a fibered strongly quasipositive knot [He3]; he also shows that for these knots, the maximal self-linking number is $\overline{sl}(K) = -\chi(K)$ [He2]. Furthermore, from the work of Etnyre and Van Horn-Morris [EV], we know that for fibered knots K in S^3 that support the standard contact structure there is a unique transversal isotopy class at $\overline{sl}$. In the present paper, all of these ideas are brought to bear on the class of iterated torus knots, and this motivates the following conjecture concerning general fibered knots:

Conjecture 1.4. *Let K be a fibered knot in S^3; then K fails the UTP if and only if $\xi_K \cong \xi_{std}$, and hence if and only if K is fibered strongly quasipositive. Moreover, if a topologically non-trivial fibered knot K fails the UTP, then it supports cablings that are transversally non-simple.*

We also ask the following question of non-fibered knots in S^3:

Question 1.5. *If K is a non-fibered strongly quasipositive knot, does K fail the UTP and support transversally non-simple cablings?*

We will be using tools developed by Giroux, Kanda, and Honda, and used by Etnyre and Honda in their work, namely convex tori and annuli, the classification of tight contact structures on solid tori and thickened tori, and the Legendrian classificaton of torus knots. Most of the results we use can be found in [H1], [EH1], [H2], or [L], and if we use a lemma, proposition, or theorem from one of these works, it will be specifically referenced.

The plan of the note is as follows. In §2 we recall definitions, notation, and identities used in [L] and [EH1]. In §3 we outline a strategy of proof of Theorem 1.1 that yields the statement of two key lemmas. In §4 and §5 we prove the first lemma. In §6 we prove the second lemma and complete the proof of Theorem 1.1. In §7 we prove Theorem 1.2.

2. Definitions, notation, and identities

2.1. **Iterated torus knots.** *Iterated torus knots*, as topological knot types, can be defined recursively. Let 1-iterated torus knots be simply torus knots (p_1, q_1) with p_1 and q_1 co-prime

nonzero integers, and $|p_1|, q_1 > 1$. Here p_1 is the algebraic intersection with a longitude, and q_1 is the algebraic intersection with a meridian in the preferred framing for a torus representing the unknot. Then for each (p_1, q_1) torus knot, take a solid torus regular neighborhood $N((p_1, q_1))$; the boundary of this is a torus, and given a framing we can describe simple closed curves on that torus as co-prime pairs (p_2, q_2), with $q_2 > 1$. In this way we obtain all 2-iterated torus knots, which we represent as ordered pairs, $((p_1, q_1), (p_2, q_2))$. Recursively, suppose the $(r-1)$-iterated torus knots are defined; we can then take regular neighborhoods of all of these, choose a framing, and form the r-iterated torus knots as ordered r-tuples $((p_1, q_1), ..., (p_{r-1}, q_{r-1}), (p_r, q_r))$, again with p_r and q_r co-prime, and $q_r > 1$.

For ease of notation, if we are looking at a general r-iterated torus knot type, we will refer to it as K_r; a Legendrian representative will usually be written as L_r. Note that we will use the letter r both for the rotation number (see below) and as an index for our iterated torus knots; context will distinguish between the two uses.

We will study iterated torus knots using two framings. The first is the standard framing for a torus, where the meridian bounds a disc inside the solid torus, and we use the preferred longitude which bounds a surface in the complement of the solid torus. We will refer to this framing as $\mathcal{C}$. The second framing is a non-standard framing using a different longitude that comes from the cabling torus. More precisely, to identify this non-standard longitude on $\partial N(K_r)$, we first look at K_r as it is embedded in $\partial N(K_{r-1})$. We take a small neighborhood $N(K_r)$ such that $\partial N(K_r)$ intersects $\partial N(K_{r-1})$ in two parallel simple closed curves. These curves are longitudes on $\partial N(K_r)$ in this second framing, which we will refer to as $\mathcal{C}'$. Note that this $\mathcal{C}'$ framing is well-defined for any cabled knot type. Moreover, for purpose of calculations there is an easy way to change between the two framings, which will be reviewed below.

Given a simple closed curve (μ, λ) on a torus, measured in some framing as having μ meridians and λ longitudes, we will say this curve has slope of $\frac{\lambda}{\mu}$; i.e., longitudes over meridians. Therefore we will refer to the longitude in the $\mathcal{C}'$ framing as ∞', and the longitude in the $\mathcal{C}$ framing as ∞. The meridian in both framings will have slope 0.

We will also use a convention that meridians in the standard $\mathcal{C}$ framing, that is, algebraic intersection with ∞, will be denoted by upper-case P's. On the other hand, meridians in the non-standard $\mathcal{C}'$ framing, that is, algebraic intersection with ∞', will be denoted by lower-case p's. Given a curve (P, q) on a torus $\partial N(K)$, there is then a relationship between the framings $\mathcal{C}'$ and $\mathcal{C}$ on $\partial N(K)$. In terms of a change of basis, we get from $\mathcal{C}'$ to $\mathcal{C}$ by multiplying on the left by the matrix $\begin{pmatrix} 1 & Pq \\ 0 & 1 \end{pmatrix}$.

Given an iterated torus knot type $K_r = ((p_1, q_1), ..., (p_r, q_r))$ where the p_i's are measured in the $\mathcal{C}'$ framing, we define two quantities. The two quantities are:

$$A_r := \sum_{\alpha=1}^{r} p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta \prod_{\beta=\alpha}^{r} q_\beta \qquad B_r := \sum_{\alpha=1}^{r} \left(p_\alpha \prod_{\beta=\alpha+1}^{r} q_\beta \right) + \prod_{\alpha=1}^{r} q_\alpha \tag{1}$$

Note here we use a convention that $\prod_{\beta=r+1}^{r} q_\beta := 1$. Also, if we restrict to the first i

iterations, that is, to $K_i = ((p_1, q_1), ..., (p_i, q_i))$, we have an associated A_i and B_i. For example, $A_i := \sum_{\alpha=1}^{i} p_\alpha \prod_{\beta=\alpha+1}^{i} q_\beta \prod_{\beta=\alpha}^{i} q_\beta$.

Finally, from [L] we obtain four useful identities which we will apply extensively throughout this note:

$$A_r = q_r^2 A_{r-1} + p_r q_r \qquad B_r = q_r B_{r-1} + p_r \qquad P_r = q_r A_{r-1} + p_r \qquad A_r = P_r q_r \tag{2}$$

2.2. **Legendrian knots, convex tori, and the UTP.** Recall that for Legendrian knots embedded in S^3 with the standard tight contact structure, there are two classical invariants of Legendrian isotopy classes, namely the Thurston-Bennequin number, *tb*, and the rotation number, r. For a given topological knot type, if the ordered pair (r, tb) completely determines the Legendrian isotopy classes, then that knot type is said to be *Legendrian simple*. For transversal knots there is one classical invariant, the self-linking number *sl*; for a given topological knot type, if the value of *sl* completely determines the transversal isotopy classes, then that knot type is said to be *transversally simple*. For a given topological knot type, if we plot Legendrian isotopy classes at points (r, tb), we obtain a plot of points that takes the form of a *Legendrian mountain range* for that knot type.

We will be examining Legendrian knots which are embedded in convex tori. Recall that the characteristic foliation induced by the contact structure on a convex torus can be assumed to have a standard form, where there are $2n$ parallel *Legendrian divides* and a one-parameter family of *Legendrian rulings*. Parallel push-offs of the Legendrian divides gives a family of $2n$ *dividing curves*, referred to as Γ. For a particular convex torus, the slope of components of Γ is fixed and is called the *boundary slope* of any solid torus which it bounds; however, the Legendrian rulings can take on any slope other than that of the dividing curves by Giroux's Flexibility Theorem [G]. A *standard neighborhood* of a Legendrian knot L will have two dividing curves and a boundary slope of $\frac{1}{tb(L)}$.

We can now state the definition of the *uniform thickness property* as given by Etnyre and Honda [EH1]. For a knot type K, define the *contact width* of K to be

$$w(K) = \sup \frac{1}{\text{slope}(\Gamma_{\partial N})} \tag{3}$$

In this equation the N are solid tori having representatives of K as their cores; slopes are measured using the preferred framing where the longitude has slope ∞; the supremum is taken over all solid tori N representing K where ∂N is convex. A knot type K then satisfies the UTP if the following hold:

1. $\overline{tb}(K) = w(K)$, where $\overline{tb}$ is the maximal Thurston-Bennequin number for K.
2. Every solid torus N representing K can be thickened to a standard neighborhood of a maximal *tb* Legendrian knot.

For a topological knot type K, if N is a solid torus having a representative of K as its core and convex boundary, then N *fails to thicken* if for all $N' \supset N$, we have $\text{slope}(\Gamma_{\partial N'}) = \text{slope}(\Gamma_{\partial N})$.

If we define t to be the twisting of the contact planes along L with respect to the $\mathcal{C}'$ framing on $\partial N(L)$, equation 2.1 in [EH1] gives us:

$$tb(L) = Pq + t(L) \tag{4}$$

Observe that $t(L)$ is also the twisting of the contact planes with respect to the framing given by ∂N, and so is equal to $-\frac{1}{2}$ times the geometric intersection number of L with $\Gamma_{\partial N}$. $\overline{t}$ will denote the maximal twisting number with respect to this framing.

We also had two definitions introduced in [L] that will be useful in this note.

Definition 2.1. Let N be a solid torus with convex boundary in standard form, and with slope$(\Gamma_{\partial N}) = \frac{a}{b}$ in some framing. If $|2b|$ is the geometric intersection of the dividing set Γ with a longitude ruling in that framing, then we will call $\frac{a}{b}$ the *intersection boundary slope.*

Note that when we have an intersection boundary slope $\frac{a}{b}$, then $2\text{gcd}(a, |b|)$ is the number of dividing curves.

Definition 2.2. For $r \geq 1$ and positive integer k, define N_r^k to be any solid torus representing K_r with intersection boundary slope of $-\frac{k+1}{A_r k + B_r}$, as measured in the $\mathcal{C}'$ framing. Also define the integer $n_r^k := \text{gcd}((k+1), (A_r k + B_r))$.

Note that N_r^k has $2n_r^k$ dividing curves. Note also that the above definition is only for $k \geq 1$. However, we will also define N_r^0 to be a standard neighborhood of a $\overline{tb}(K_r)$ representative, and thus have this as the $k = 0$ case.

Finally, recall that if $\mathcal{A}$ is a convex annulus with Legendrian boundary components, then dividing curves are arcs with endpoints on either one or both of the boundary components. Dividing curves that are boundary parallel are called *bypasses*; an annulus with no bypasses is said to be *standard convex*.

2.3. **Universally tight contact structures.** Recall that a contact structure ξ on a 3-manifold M is said to be *overtwisted* if there exists an embedded disc D which is tangent to ξ everywhere along ∂D, and a contact structure is *tight* if it is not overtwisted. Moreover, one can further analyze tight contact 3-manifolds (M, ξ) by looking at what happens to ξ when pulled back to the universal cover $\widetilde{M}$ via the covering map $\pi : \widetilde{M} \to M$. In particular, if the pullback of ξ remains tight, then (M, ξ) is said to be *universally tight*.

The classification of universally tight contact structures on solid tori is known from the work of Honda. Specifically, from Proposition 5.1 in [H1], we know there are exactly two universally tight contact structures on $S^1 \times D^2$ with boundary torus having two dividing curves and slope $s < -1$ in some framing. These are such that a convex meridional disc has boundary-parallel dividing curves that separate half-discs all of the same sign, and thus the two contact structures differ by $-id$. (If $s = -1$, there is only one tight contact structure, and it is universally tight.)

Also from the work of Honda, we know that if ξ is a contact structure which is everywhere transverse to the fibers of a circle bundle M over a closed oriented surface Σ, then ξ is universally tight. This is the content of Lemma 3.9 in [H2], and such a transverse contact structure is said to be *horizontal*.

2.4. **Transverse push-offs of Legendrian knots.** Given a Legendrian knot L, recall that there are well-defined *positive and negative transverse push-offs*, denoted by $T_+(L)$ and $T_-(L)$, respectively. Moreover, the self-linking numbers of these transverse push-offs are given by the formula

$$sl(T_{\pm}(L)) = tb(L) \mp r(L)$$

3. Strategy of proof for Theorem 1.1

In this section we present a strategy of proof for Theorem 1.1. We begin with a theorem that in previous works has in effect been proved, but not stated. In this theorem K is a knot type and $K_{(P,q)}$ is the (P,q)-cabling of K.

Theorem 3.1 (Etnyre-Honda, L.). *If K satisfies the UTP, then $K_{(P,q)}$ also satisfies the UTP.*

Proof. The case where the cabling fraction $\frac{P}{q} < w(K)$ is the content of Theorem 1.3 in [EH1]. For the case where $\frac{P}{q} > w(K)$, the proof follows from examining the proofs of Theorem 3.2 [EH1] and Theorem 1.1 in [L] and observing that Legendrian simplicity of K is not needed to preserve the UTP. □

With this theorem in mind, we will prove Theorem 1.1 by way of two lemmas, one of which uses induction. For this purpose we make the following inductive hypothesis, which from here on we will refer to as *the inductive hypothesis*. We will need to justify its veracity for the base case of positive torus knots.

Inductive hypothesis: Let $K_r = ((P_1, q_1), ..., (P_r, q_r))$ be an iterated torus knot, as measured in the standard $\mathcal{C}$ framing. Assume that the following hold:

1. $P_i > 0$ for all i, where $1 \leq i \leq r$. (Thus $A_i > 0$ for all i as well.)
2. $0 < \overline{tb}(K_r) = w(K_r) \leq A_r$. (Thus $-A_r < \overline{t}(K_r) \leq 0$.)
3. Any solid torus N_r representing K_r thickens to some N_r^k (including N_r^0 which is a standard neighborhood of a $\overline{tb}$ representative).
4. If N_r fails to thicken then it is an N_r^k, and it has at least $2n_r^k$ dividing curves.
5. The candidate non-thickenable N_r^k exist and actually fail to thicken for $k \geq C_r$, where C_r is some positive integer that varies according to the knot type K_r. Moreover, these N_r^k that fail to thicken have contact structures that are universally tight, with convex meridian discs containing bypasses all of the same sign. Also, a Legendrian ruling preferred longitude on these ∂N_r^k has rotation number zero for $k > 0$.

Our first key lemma used in proving Theorem 1.1 is the following, which along with the base case of positive torus knots, will show that if $K_r = ((P_1, q_1), ..., (P_r, q_r))$ is such that $P_i > 0$ for all i, then K_r fails the UTP.

Lemma 3.2. *Suppose K_r satisfies the inductive hypothesis, and K_{r+1} is a cabling where $P_{r+1} > 0$; then K_{r+1} satisfies the inductive hypothesis.*

The main idea in the argument used to prove this lemma will be that since K_r satisfies the inductive hypothesis, there is an infinite collection of non-thickenable solid tori whose boundary slopes form an increasing sequence converging to $-\frac{1}{A_r}$ in the $\mathcal{C}'$ framing (which is ∞ in the $\mathcal{C}$ framing). As a consequence, it will be shown that cabling slopes with $P_{r+1} > 0$ in the $\mathcal{C}$ framing will have a similar sequence of non-thickenable solid tori.

Our second key lemma is the following, which along with Theorem 3.1 and the fact that negative torus knots satisfy the UTP, will show that if at least one of the $P_i < 0$, then K_r satisfies the UTP.

Lemma 3.3. *Suppose K_r satisfies the inductive hypothesis, and K_{r+1} is a cabling where $P_{r+1} < 0$; then K_{r+1} satisfies the UTP.*

Our outline for the next three sections is as follows. In the next section, §4, we establish the truth of the inductive hypothesis for the base case of positive torus knots. In §5 we prove Lemma 3.2, and in §6 we prove Lemma 3.3.

4. Positive torus knots fail the UTP

In this section we show that positive torus knots (p_1, q_1) satisfy the inductive hypothesis described in §3. From Lemma 4.3 in [L], we know that items 1-4 of the inductive hypothesis are satisfied; it remains to establish item 5, that each solid torus candidate N_1^k actually exists with a universally tight contact structure and the appropriate complement in S^3, and that these N_1^k indeed fail to thicken (for all $k \geq 0$ in this case of positive torus knots).

To establish item 5, we employ arguments similar to those used in [EH1] for solid tori representing the $(2, 3)$ torus knot, specifically Lemmas 5.2 and 5.3 in [EH1]. From Lemma 4.3 in [L], we know that if N_1^k fails to thicken, its complement $M_1^k := S^3 \backslash N_1^k$ must be contactomorphic to the manifold obtained by taking a neighborhood of a Hopf link $N(L_1) \sqcup N(L_2)$ and a standard convex annulus $\mathcal{A}$ joining the two neighborhoods of the Hopf link, where $\mathcal{A}$ has boundary components that are Legendrian ruling representatives of $K_1 = (p_1, q_1)$. Moreover, we know that the two components of the Hopf link must have tb values equal to $-(p_1k + 1)$ and $-(q_1k + 1)$, respectively, for $k \geq 0$.

We first show that the candidate N_1^k have universally tight contact structures.

Lemma 4.1. *If N_1^k fails to thicken, then its contact structure is universally tight; moreover, for $k > 0$, a convex meridian disc contains bypasses that all bound half-discs of the same sign. Also, a Legendrian ruling preferred longitude on ∂N_1^k has rotation number zero for $k > 0$.*

Proof. The lemma is immediately true for $k = 0$, so we may assume that $k > 0$. To fix notation, let L_1 be the Legendrian unknot with $tb = -(p_1k + 1)$ and let L_2 be the unknot with $tb = -(q_1k + 1)$. Then $N(L_1)$ thickens outward to $S^3 \backslash N(L_2)$; we denote $T_1 := \partial N(L_1)$ and $T_2 := \partial(S^3 \backslash N(L_2))$. Since T_1 and T_2 are convex, we can take $[0, 1]$-invariant neighborhoods of each; our convention will be that the two $T_i \times \{0\}$ will bound a thickened torus that contains the two $T_i \times \{1\}$.

Now $T_2 \times \{0\}$ is a convex torus with dividing curves that divide the torus into two annuli, $\mathcal{A}_+$ and $\mathcal{A}_-$. We locate a (topological) meridian curve μ on $T_2 \times \{0\}$ that intersects each dividing curve efficiently $(q_1k + 1)$ times, and so that $\mu \backslash \partial\mathcal{A}$ consists of q_1 arcs which intersect $\mathcal{A}_+$ and $\mathcal{A}_-$ at least k times each. We then can realize μ as a Legendrian ruling using Theorem 3.7 in [H1].

We then examine a horizontal convex annulus $\mathcal{A}_H$ in the space $(S^3 \backslash N(L_2)) \backslash N(L_1)$, bounded by meridian rulings on $T_i \times \{0\}$. This horizontal convex annulus $\mathcal{A}_H$ has two dividing curves that connect its two boundary components; the other q_1k bypasses have endpoints on $T_2 \times \{0\}$. By Lemma 4.14 in [H1], we may assume that all of these bypasses

are boundary compressible, meaning there are no nested bypasses. The two dividing curves connecting the two boundary components of $\mathcal{A}_H$ thus divide $\mathcal{A}_H$ into two discs, one containing all bypasses of positive sign, the other disc containing all negative bypasses. We will show that in fact all bypasses on $\mathcal{A}_H$ must be of the same sign.

To this end, let $\mathcal{A}$ be the standard convex annulus with (p_1, q_1) Legendrian rulings as its boundary components on $T_i \times \{1\}$. We first examine $\mathcal{A} \cap \mathcal{A}_H$, which is q_1 arcs with endpoints on the two $T_i \times \{1\}$. At first glance, it is possible that there may be points of intersection between these q_1 arcs and the boundary-parallel dividing curves on $\mathcal{A}_H$. However, up to a choice of contact vector field for the convex annulus $\mathcal{A}_H$, we may assume that all boundary-parallel dividing curves for $\mathcal{A}_H$ are in a collar neighborhood of $T_2 \times \{0\}$ and avoid $\mathcal{A}$. This contact vector field may also be chosen so that the two non-separating dividing curves on $\mathcal{A}_H$ intersect $\mathcal{A}$ transversely.

Now $(T_2 \times \{1\}) \backslash \partial\mathcal{A}$ is one of the annuli that forms ∂N_1^k, and the intersection of this annulus with $\mathcal{A}_H$ will be q_1 arcs, which we denote as γ_j for $1 \leq j \leq q_1$. By the above considerations we thus have that, as a collection, the γ_j have support that intersects all of the $q_1 k$ bypasses on $\mathcal{A}_H$. See Figure 1.

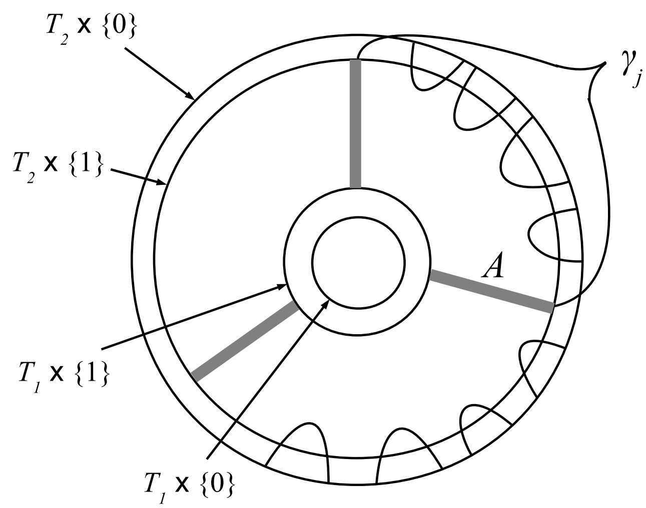

FIGURE 1. Shown is the horizontal convex annulus $\mathcal{A}_H$. The thick gray arcs represent the intersection of $\mathcal{A}$ with $\mathcal{A}_H$. Some of the $q_1 k$ bypasses are shown; in the figure, $q_1 = 3$.

We next perform edge-rounding for the curves of intersection of $\partial N(\mathcal{A})$ and the two annuli coming from $(T_i \times \{1\}) \backslash \partial\mathcal{A}$; after edge-rounding we obtain ∂N_1^k. Thus ∂N_1^k intersects $\mathcal{A}_H$ in q_1 (topological) meridian curves for N_1^k; this set of curves, call it C, is *nonisolating* in the sense of section 3.3.1 in [H1], meaning that each curve is transverse to the dividing set of $\mathcal{A}_H$ and every component of $\mathcal{A}_H \backslash (\Gamma \cup C)$ has boundary that intersects Γ. Moreover, C is also a nonisolating set of curves on ∂N_1^k. Then by Theorem 3.7 in [H1], we can realize the q_1 topological meridian curves as Legendrian meridian curves for ∂N_1^k.

Now these Legendrian meridian curves may not have efficient geometric intersection with $\Gamma_{\partial N_1^k}$. However, by the construction of ∂N_1^k, any holonomy of dividing curves on the two annuli coming from the two sides of $N(\mathcal{A})$ cancels each other out. Thus we can destabilize these Legendrian meridian curves on the surface ∂N_1^k so that they do have

geometric intersection $2(k+1)$ with $\Gamma_{\partial N_1^k}$, and we can do so *away from the* γ_j. These destabilizations can thus be accomplished by attachment of bypasses off of the q_1 convex meridian discs, but these (attached) bypasses will avoid the original $q_1 k$ bypasses along the γ_j. The resulting q_1 convex meridian discs therefore inherit the bypasses of $\mathcal{A}_H$.

By construction, it is possible that one of the q_1 convex meridian discs may inherit $k+1$ bypasses from $\mathcal{A}_H$; if this is the case, however, these bypasses must all be of the same sign, and we have the desired conclusion. So we may assume that each of the q_1 meridian discs intersects k bypasses of $\mathcal{A}_H$. So suppose, for contradiction, that the $q_1 k$ bypasses on $\mathcal{A}_H$ have mixed sign, meaning some are negative and some are positive. Since each of the q_1 meridian discs is a convex meridian disc for N_1^k, then by the classification of tight contact structures on solid tori we know that if one of the discs has a negative bypass, then all of them must; the same is true for positive bypasses. But since the negative bypasses on $\mathcal{A}_H$ are grouped in succession, and since we may assume $q_1 \geq 3$, this forces one of the discs to inherit only negative bypasses, contradicting the fact that it is supposed to also have positive bypasses. Thus all of the bypasses on $\mathcal{A}_H$ must be of the same sign, as must be all of the bypasses on a convex meridian disc for N_1^k. As a result the contact structure for N_1^k is universally tight.

We can now calculate the rotation number for the (p_1, q_1) ruling on $N(L_1)$. Since $(S^3 \backslash N(L_2)) \backslash N(L_1)$ is universally tight, one can show that if Σ_1 is a convex Seifert surface for the longitude on $\partial N(L_1)$, we must have $r(\partial \Sigma_1) = \pm(p_1 k)$. By Lemma 2.2 in [EH1], we have that the (p_1, q_1) ruling on $\partial N(L_1)$ has rotation number equal to

$$r((p_1, q_1)) = p_1 r(\partial D_1) + q_1 r(\partial \Sigma_1) \tag{5}$$

This yields $r((p_1, q_1)) = \pm(p_1 q_1 k)$. We now let D be a meridian disc for N_1^k and Σ be a Seifert surface for the preferred longitude on ∂N_1^k. We know $r(\partial D) = \pm k$, and we know that the (p_1, q_1) torus knot, which is ∞' on ∂N_1^k, is actually a $(p_1 q_1, 1)$ knot on ∂N_1^k in the preferred framing. So using a similar equation from above, we obtain that $r((p_1 q_1, 1)) = \pm(p_1 q_1 k) = \pm(p_1 q_1 k) + q_1 r(\partial \Sigma)$. Thus $r(\partial \Sigma) = 0$. □

We note that there are two universally tight contact structures, diffeomorphic by $-id$, which satisfy the conditions set by the above lemma. We now show that these appropriate N_1^k and associated M_1^k actually exist in S^3.

Lemma 4.2. *The standard tight contact structure on S^3 splits into a universally tight contact structure on N_1^k and M_1^k.*

Proof. The idea is to build S^3. To begin, choose one of the above two universally tight candidates for N_1^k. We then claim we can join N_1^k to itself by a standard convex annulus $\mathcal{A}'$ with boundary ∞' rulings so that $R := N_1^k \cup N(\mathcal{A}')$ is a (universally tight) thickened torus with boundary $T_2 - T_1$ having associated boundary slopes of $-\frac{q_1 k+1}{1}$ and $-\frac{1}{p_1 k+1}$ and two dividing curves. One way to see this is that we can think of ∂N_1^k as being composed of four annuli, one from $T_2 \backslash \mathcal{A}'$, one from $-T_1 \backslash \mathcal{A}'$, and two from $\partial \mathcal{A}' \times [-\epsilon, \epsilon]$. Since we are constructing the thickened torus, with a suitable choice of holonomy of $\mathcal{A}'$, we can assure that the dividing curves on $-T_1$ have only one longitude, and two components. Since we know the twisting of ∞' on N_1^k is equal to $-(p_1 q_1 k + p_1 + q_1)$, a calculation shows that the dividing curves on $-T_1$ must have slope $-\frac{1}{p_1 k+1}$. But then the slopes of the dividing

curves on $-T_1$ and ∂N_r^k are determined, making the slope of dividing curves on T_2 equal to $-\frac{q_1k+1}{1}$ based on equation 8 in Lemma 4.3 in [L].

Now as in the proof of Lemma 5.2 in [EH1], the contact structure on $N_1^k \cup N(\mathcal{A}')$ can be isotoped to be transverse to the fibers of $N_1^k \cup N(\mathcal{A}')$, which are parallel copies of K_1, while preserving the dividing set on $\partial(N_1^k \cup N(\mathcal{A}'))$. Such a horizontal contact structure is universally tight.

We then use the classification of tight contact structures on S^3, solid tori, and thickened tori to conclude that any tight contact structure on $R = T^2 \times [1,2]$ with boundary conditions being tori with two dividing curves and slopes $-\frac{q_1k+1}{1}$ and $-\frac{1}{p_1k+1}$ glues together with standard neighborhoods of unknots with those boundary slopes to give the tight contact structure on S^3. $\square$

We now show that these N_1^k with complements M_1^k fail to thicken.

Lemma 4.3. *The N_1^k with complement M_1^k fail to thicken.*

Proof. By inequality 14 in [L], it suffices to show that N_1^k does not thicken to any $N_1^{k'}$ for $k' < k$. So to this end, observe that the (p_1, q_1) positive torus knot is a fibered knot over S^1 with fiber a Seifert surface Σ of genus $g = \frac{(p_1-1)(q_1-1)}{2}$ (see [Mi]). Moreover, the monodromy is periodic with period p_1q_1. Thus, M_1^k has a p_1q_1-fold cover $\widetilde{M_1^k} \cong S^1 \times \Sigma$. If one thinks of M_1^k as $\Sigma \times [0,1]$ modulo the relation $(x,0) \sim (\phi(x),1)$ for monodromy ϕ, then one can view $\widetilde{M_1^k}$ as p_1q_1 copies of $\Sigma \times [0,1]$ cyclically identified via the same monodromy. Now note that downstairs in M_1^k, ∞' intersects any given Seifert surface p_1q_1 times efficiently. It is therefore evident that we can view M_1^k as a Seifert fibered space with base space Σ and two singular fibers (the components of the Hopf link). The regular fibers are topological copies of ∞', which itself is a Legendrian ruling on ∂N_1^k with twisting $-(A_1k + B_1)$. In fact, the regular fibers can be assumed to be Legendrian isotopic to the ∂N_1^k-fibers except for small neighborhoods around the singular fibers.

We claim the pullback of the tight contact structure to $\widetilde{M_1^k}$ admits an isotopy where the S^1 fibers are all Legendrian and have twisting number $-(A_1k + B_1)$ with respect to the product framing. This isotopy can be accomplished because in $\widetilde{M_1^k}$, the lifts of the singular fibers have tight neighborhoods with convex boundary tori which have dividing curves with one longitude and where ∞' has twisting $-(A_1k + B_1)$. Thus these neighborhoods of the lifts of the singular fibers are in fact standard neighborhoods of a Legendrian fiber with twisting $-(A_1k + B_1)$; the contact structure can then be isotoped so that every fiber inside these neighborhoods is Legendrian with twisting $-(A_1k + B_1)$.

So, if N_1^k can be thickened to $N_1^{k'}$, then there exists a Legendrian curve topologically isotopic to the regular fiber of the Seifert fibered space M_1^k with twisting number greater than $-(A_1k + B_1)$, measured with respect to the Seifert fibration. Pulling back to the p_1q_1-fold cover $\widetilde{M_1^k}$, we have a Legendrian knot which is topologically isotopic to a fiber but has twisting greater than $-(A_1k + B_1)$. We will obtain a contradiction, thus proving that N_1^k cannot be thickened to $N_1^{k'}$.

To obtain our contradiction, we let $\pi : S^1 \times \Sigma \to \Sigma$ be the projection map onto the base space. Thus the hypothesis that N_1^k can be thickened to $N_1^{k'}$ yields a knot $\gamma \subset S^1 \times \Sigma$ which is isotopic to $\pi^{-1}(p_0)$ for some $p_0 \in \Sigma$, but where $t(\gamma) > -(A_1k + B_1)$. Thus there is a continuous isotopy $F : S^1 \times I \to S^1 \times \Sigma$ where $F(S^1 \times \{0\}) = \pi^{-1}(p_0)$ and $F(S^1 \times \{1\}) = \gamma$.

Now look at $\pi\circ F : S^1\times I \to \Sigma$. Then this is a continous map, and since $\pi\circ F(S^1\times\{0\}) = p_0$, we actually obtain $\pi\circ F : D^2 \to \Sigma$. This means that γ is contained inside a tight $S^1\times D^2$ that is fibered by Legendrian fibers with twisting $-(A_1k + B_1)$, and is thus a solid torus neighborhood of a Legendrian knot with twisting $-(A_1k + B_1)$. By the classification of tight contact structures on solid tori, such a γ cannot exist. This is our contradiction. □

5. Positive cablings that fail the UTP

Now that we know that the base case holds for positive torus knots, we begin to prove Lemma 3.2 – for the bulk of this section we will thus have that $P_{r+1} > 0$, K_r satisfies the inductive hypothesis, and we work to show that K_{r+1} satisfies the inductive hypothesis. We will need to break the proof of Lemma 3.2 into two cases, Case I being where $\frac{P_{r+1}}{q_{r+1}} > w(K_r)$, and Case II being where $w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0$. However, before we do that, we prove two general lemmas concerning iterated cablings that begin with positive torus knots.

Lemma 5.1. *If K_r is an iterated torus knot with $P_1 > 0$, then $A_r > B_r$.*

Proof. We use induction. $A_1 > B_1$ is evident from equation 1 above. Then inductively, $A_r = q_r^2A_{r-1} + p_rq_r > q_rA_{r-1} + p_r > q_rB_{r-1} + p_r = B_r$. □

We now use the above lemma to prove the following.

Lemma 5.2. *Let K_r be an iterated torus knot with $P_1 > 0$. If $k_1 < k_2$ and both $(A_rk_1 + B_r), (A_rk_2 + B_r) > 0$, then $-\frac{k_1+1}{A_rk_1+B_r} < -\frac{k_2+1}{A_rk_2+B_r}$.*

Proof. We have that $-\frac{k_1+1}{A_rk_1+B_r} < -\frac{k_2+1}{A_rk_2+B_r}$ if and only if $(k_1 + 1)(A_rk_2 + B_r) > (k_2 + 1)(A_rk_1 + Br)$. But this is true if and only if $(A_r - B_r)k_2 > (A_r - B_r)k_1$, which is true. □

We now directly address the two different cases in two different subsections.

5.1. Case I: $\frac{P_{r+1}}{q_{r+1}} > w(K_r)$.

We work through proving items 2-5 in the inductive hypothesis via a series of lemmas. The following lemma begins to address item 2.

Lemma 5.3. *If $\frac{P_{r+1}}{q_{r+1}} > w(K_r)$, then $\overline{tb}(K_{r+1}) = A_{r+1} - (P_{r+1} - q_{r+1}w(K_r)) > 0$.*

Proof. The proof is similar to that of Lemma 3.3 in [EH1]. We first claim that $\overline{t}(K_{r+1}) < 0$. If not, there exists a Legendrian L_{r+1} with $t(L_{r+1}) = 0$ and a solid torus N_r with L_{r+1} as a Legendrian divide. But then we would have a boundary slope of $\frac{P_{r+1}}{q_{r+1}} > w(K_r)$ in the $\mathcal{C}$ framing, which cannot occur.

So since $\overline{t}(K_{r+1}) < 0$, any Legendrian L_{r+1} must be a ruling on a convex ∂N_r with slope $s \geq \frac{1}{\overline{t}(K_r)}$ in the $\mathcal{C}'$ framing. But then if $s = -\frac{\lambda}{\mu} > \frac{1}{\overline{t}(K_r)}$, we have that $t(L_r) = -(p_{r+1}\lambda + q_{r+1}\mu) < -\lambda(p_{r+1} - \overline{t}(K_r)q_{r+1}) \leq -(p_{r+1} - \overline{t}(K_r)q_{r+1})$. This shows that $\overline{tb}(K_{r+1})$ is achieved by a Legendrian ruling on a convex torus having slope $\frac{1}{w(K_r)}$ in the standard $\mathcal{C}$ framing.

Finally, note that $A_{r+1} - (P_{r+1} - q_{r+1}w(K_r)) = A_{r+1} - (q_{r+1}(A_r - w(K_r)) + p_{r+1}) > A_{r+1} - (q_{r+1}^2A_r + p_{r+1}q_{r+1}) = 0$. □

With the following lemma we prove that items 3 and 4 of the inductive hypothesis hold for K_{r+1}.

Lemma 5.4. *If $\frac{p_{r+1}}{q_{r+1}} > w(K_r)$, let N_{r+1} be a solid torus representing K_{r+1}, for $r \geq 1$. Then N_{r+1} can be thickened to an $N_{r+1}^{k'}$ for some nonnegative integer k'. Moreover, if N_{r+1} fails to thicken, then it has the same boundary slope as some $N_{r+1}^{k'}$, as well as at least $2n_{r+1}^{k'}$ dividing curves.*

Proof. In this case, for the $\mathcal{C}'$ framing, we have either $p_{r+1} > 0$ or $\frac{q_{r+1}}{p_{r+1}} < \frac{1}{\overline{t}(K_r)}$ (the latter being relevant only if $\overline{t}(K_r) < 0$). The proof in this case is nearly identical to the proof of Lemma 4.4 in [L]; we will include some of the details, however, as certain particular calculations differ. Moreover, we will use modifications of this argument in Case II and thus will be able to refer to the details here.

Let N_{r+1} be a solid torus representing K_{r+1}. Let L_r be a Legendrian representative of K_r in $S^3 \backslash N_{r+1}$ and such that we can join $\partial N(L_r)$ to ∂N_{r+1} by a convex annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$ whose boundaries are (p_{r+1}, q_{r+1}) and ∞' rulings on $\partial N(L_r)$ and ∂N_{r+1}, respectively. Then topologically isotop L_r in the complement of N_{r+1} so that it maximizes tb over all such isotopies; this will induce an ambient topological isotopy of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, where we still can assume $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is convex. A picture is shown in (a) in Figure 2. In the $\mathcal{C}'$ framing we will have slope$(\Gamma_{\partial N(L_r)}) = -\frac{1}{m}$ where $m \geq 0$, since $\overline{t}(K_r) \leq 0$. Now if $m = \overline{t}(K_r)$, then there will be no bypasses on the $\partial N(L_r)$-edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, since the (p_{r+1}, q_{r+1}) ruling would be at maximal twisting. On the other hand, if $m < \overline{t}(K_r)$, then there will still be no bypasses on the $\partial N(L_r)$-edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, since such a bypass would induce a destabilization of L_r, thus increasing its tb by one – see Lemma 4.4 in [H1]. To satisfy the conditions of this lemma, we are using the fact that either $p_{r+1} > 0$ or $\frac{q_{r+1}}{p_{r+1}} < \frac{1}{\overline{t}(K_r)}$. Furthermore, we can thicken N_{r+1} through any bypasses on the ∂N_{r+1}-edge, and thus assume $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex.

Now let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. Inductively we can thicken N_r to an N_r^k with intersection boundary slope $-\frac{k+1}{A_r k + B_r}$ where k is minimized over all such thickenings (if we have $k = 0$, then we will have N_{r+1} thickening to a standard neighborhood of a knot at $\overline{tb}$ – see the proof of Theorem 1.1 in [L]; so we can assume $k > 0$). Then consider a convex annulus $\widetilde{\mathcal{A}}$ from $\partial N(L_r)$ to ∂N_r^k, such that $\widetilde{\mathcal{A}}$ is in the complement of N_r and $\partial \widetilde{\mathcal{A}}$ consists of (p_{r+1}, q_{r+1}) rulings. A picture is shown in (b) in Figure 2. By an argument identical to that used in Lemma 4.4 in [L], $\widetilde{\mathcal{A}}$ is standard convex; in brief, if $\widetilde{\mathcal{A}}$ was not standard convex, either a bypass would occur on its $\partial N(L_r)$-edge, or k would not be minimized, neither of which is true.

Now four annuli compose the boundary of a solid torus $\widetilde{N}_{r+1}$ containing N_{r+1}: the two sides of a thickened $\widetilde{\mathcal{A}}$; $\partial N_r^k \backslash \partial \widetilde{\mathcal{A}}$; and $\partial N(L_r) \backslash \partial \widetilde{\mathcal{A}}$. We can compute the intersection boundary slope of this solid torus. To this end, recall that slope$(\Gamma_{\partial N(L_r)}) = -\frac{1}{m}$ where $m > 0$ ($m = 0$ would be the $\overline{t}$ case which we have take care of above). To determine m we note that the geometric intersection of (p_{r+1}, q_{r+1}) with Γ on ∂N_r^k and $\partial N(L_r)$ must be equal, yielding the equality

$$p_{r+1} + m q_{r+1} = p_{r+1} k + p_{r+1} + q_{r+1}(A_r k + B_r) \tag{6}$$

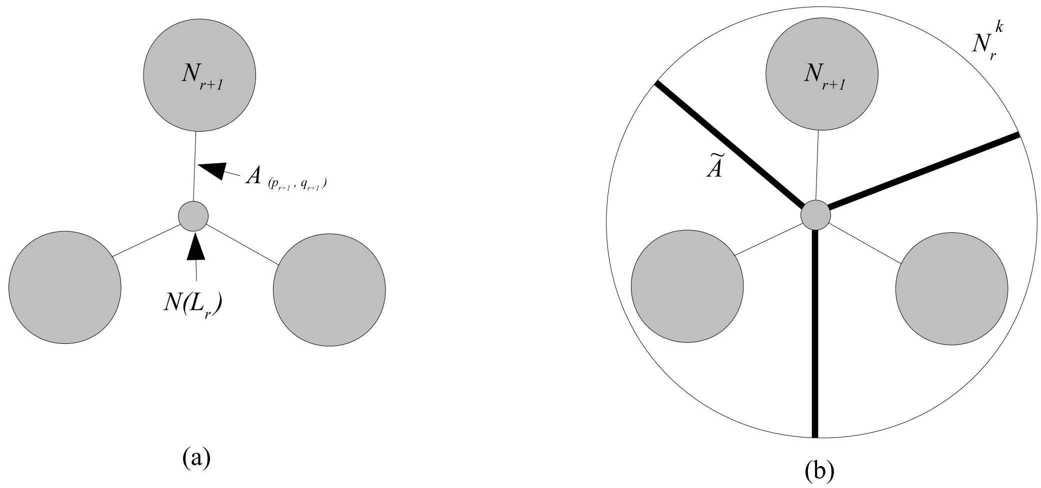

FIGURE 2. N_{r+1} is the larger solid torus in gray; $N(L_r)$ is the smaller solid torus in gray.

These equal quantities are greater than zero, since $\frac{q_{r+1}}{p_{r+1}} < -\frac{1}{m}$ – we note here that this will yield $(A_{r+1}k' + B_{r+1}) > 0$ for the calculations below. In the meantime, however, the above equation gives

$$m = p_{r+1}\frac{k}{q_{r+1}} + A_r k + B_r \tag{7}$$

We define the integer $k' := \frac{k}{q_{r+1}}$. We now choose (p'_{r+1}, q'_{r+1}) to be a curve on these two tori such that $p_{r+1}q'_{r+1} - p'_{r+1}q_{r+1} = 1$, and we change coordinates to a framing $\mathcal{C}''$ via the map $((p_{r+1}, q_{r+1}), (p'_{r+1}, q'_{r+1})) \mapsto ((0,1), (-1,0))$. Under this map we obtain

$$\text{slope}(\Gamma_{\partial N_r^k}) = \frac{q'_{r+1}(A_r k + B_r) + p'_{r+1}(q_{r+1}k' + 1)}{A_{r+1}k' + B_{r+1}} \tag{8}$$

$$\text{slope}(\Gamma_{\partial N(L_r)}) = \frac{q'_{r+1}(p_{r+1}k' + A_r k + B_r) + p'_{r+1}}{A_{r+1}k' + B_{r+1}} \tag{9}$$

We then obtain in the $\mathcal{C}'$ framing, after edge-rounding, that the intersection boundary slope of $\widetilde{N}_{r+1}$ is

$$\begin{aligned} \text{slope}(\Gamma_{\partial \widetilde{N}_{r+1}}) &= \frac{q'_{r+1}(A_r k + B_r) + p'_{r+1}(q_{r+1}k' + 1)}{A_{r+1}k' + B_{r+1}} \\ &- \frac{q'_{r+1}(p_{r+1}k' + A_r k + B_r) + p'_{r+1}}{A_{r+1}k' + B_{r+1}} \\ &- \frac{1}{A_{r+1}k' + B_{r+1}} \\ &= -\frac{k' + 1}{A_{r+1}k' + B_{r+1}} \end{aligned} \tag{10}$$

This shows that any N_{r+1} representing K_{r+1} can be thickened to one of the $N^{k'}_{r+1}$, and if N_{r+1} fails to thicken, then it has the same boundary slope as some $N^{k'}_{r+1}$. We now note that if N_{r+1} fails to thicken, and if it has the minimum number of dividing curves over all such N_{r+1} which fail to thicken and have the same boundary slope as $N^{k'}_{r+1}$, then N_{r+1} is actually an $N^{k'}_{r+1}$, by an argument identical to that used in Lemma 4.4 in [L]. In brief, if N_{r+1} fails to thicken and is at minimum number of dividing curves, then taking $N_{r+1} \cup N(L_r) \cup (\mathcal{A}_{(p_{r+1},q_{r+1})})$ gives an N^k_r; one then concludes that N_{r+1} is an $N^{k'}_{r+1}$. □

We now finish the proof of item 2 of the inductive hypothesis.

Lemma 5.5. *If $\frac{p_{r+1}}{q_{r+1}} > w(K_r)$, then $w(K_{r+1}) = \overline{tb}(K_{r+1})$.*

Proof. We show that $\frac{1}{\overline{t}(K_{r+1})} < -\frac{k'+1}{A_{r+1}k'+B_{r+1}}$ for any candidate $N^{k'}_{r+1}$. As a consequence, since any N_{r+1} thickens to some $N^{k'}_{r+1}$ (including $k'=0$), we have, to prevent overtwisting, that $w(K_{r+1}) = \overline{tb}(K_{r+1})$. Now note that our intended inequality is automatically true if $\overline{t}=0$; thus we may assume that $\overline{t}(K_{r+1}) < 0$.

We have that $\frac{1}{\overline{t}(K_{r+1})} < -\frac{k'+1}{A_{r+1}k'+B_{r+1}}$ holds if and only if

$$A_{r+1}k' + B_{r+1} > (k'+1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) \tag{11}$$

Inductively we know that $\frac{1}{\overline{t}(K_r)} < -\frac{k+1}{A_r k+B_r}$ where $k = k'q_{r+1}$. This implies that

$$A_r k + B_r > -(k'q_{r+1}+1)\overline{t}(K_r) \tag{12}$$

We can now prove inequality 11; we begin with $A_{r+1}k' + B_{r+1}$. We have:

$$\begin{aligned}
A_{r+1}k' + B_{r+1} &= (q^2_{r+1}A_r + q_{r+1}p_{r+1})k' + p_{r+1} + q_{r+1}B_r \\
&= q_{r+1}(A_r k + B_r) + p_{r+1}q_{r+1}k' + p_{r+1} \\
&> -q_{r+1}(k'q_{r+1}+1)\overline{t}(K_r) + p_{r+1}q_{r+1}k' + p_{r+1} \\
&= (k'+1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) + k'(q_{r+1}-1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) \\
&> (k'+1)(p_{r+1} - q_{r+1}\overline{t}(K_r)) \qquad (13)
\end{aligned}$$

□

We conclude this subsection by proving item 5 of the inductive hypothesis.

Lemma 5.6. *If $\frac{p_{r+1}}{q_{r+1}} > w(K_r)$, the candidate $N^{k'}_{r+1}$ exist and actually fail to thicken for $k' \geq C_{r+1}$, where C_{r+1} is some positive integer. Moreover, these $N^{k'}_{r+1}$ have contact structures that are universally tight and have convex meridian discs whose bypasses bound half-discs all of the same sign. Also, the preferred longitude on $\partial N^{k'}_{r+1}$ has rotation number zero for $k' > 0$.*

Proof. We first prove that the contact structure on a candidate $N^{k'}_{r+1}$ which fails to thicken is universally tight. To see this note that from Lemma 5.2 above, and the inductive hypothesis, such a candidate $N^{k'}_{r+1}$ is embedded inside a N^k_r with a universally tight contact structure. Now there is a q_{r+1}-fold cover of N^k_r that maps q_{r+1} lifts $\widetilde{N}^{k'}_{r+1}$ to $N^{k'}_{r+1}$, the lifts

themselves each being an $S^1 \times D^2$. This cover in turn has a universal cover $\mathbb{R} \times D^2$ that contains q_{r+1} copies of a universal cover $\mathbb{R} \times D^2$ of $N^{k'}_{r+1}$. Since, by the inductive hypothesis, the universal cover of N^k_r has a tight contact structure, a tight contact structure is thus induced on the universal cover of $N^{k'}_{r+1}$.

To see that a meridian disc for $N^{k'}_{r+1}$ contains bypasses all of the same sign, note that this is immediate if $\partial N^{k'}_{r+1}$ has two dividing curves. For the case of $2n$ dividing curves where $n > 1$, we argue in a similar fashion to Lemma 4.1. Specifically, since a meridian disc for N^k_r inductively has bypasses all of the same sign, a horizontal annulus $\mathcal{A}_H$ with boundary on ∂N^k_r and $\partial N(L_r)$ will have $q_r k'$ bypasses all of the same sign. Thus, as in Lemma 4.1, a meridian disc for $N^{k'}_{r+1}$ will inherit $k'+1$ bypasses all of the same sign.

To show that the preferred longitude on $\partial N^{k'}_{r+1}$ has rotation number zero, we use an argument similar to that used in Lemma 4.1. We call the meridian disc for N^k_r, D_r, and the Seifert surface for the preferred longitude on ∂N^k_r, Σ_r. If we then look at the (P_{r+1}, q_{r+1}) cable on ∂N^k_r, we can calculate its rotation number as

$$r((P_{r+1}, q_{r+1})) = P_{r+1} r(\partial D_r) + q_{r+1} r(\partial \Sigma_r) = P_{r+1}(\pm q_{r+1} k')$$

But then since this same knot is a $(P_{r+1}q_{r+1}, 1)$ cable on $\partial N^{k'}_{r+1}$, we have that $r((P_{r+1}, q_{r+1}))$ $= P_{r+1}q_{r+1}(\pm k') + q_{r+1} r(\partial\Sigma)$, where Σ is a Seifert surface for the preferred longitude on $\partial N^{k'}_{r+1}$. This implies that $r(\partial\Sigma) = 0$.

Now we know inductively that there exists a C_r such that if $k \geq C_r$, then the N^k_r exist and fail to thicken. So suppose $k/q_{r+1} \in \mathbb{N}$ for some $k \geq C_r$. We will show that $N^{k'}_{r+1}$ exists and fails to thicken for $k' := k/q_{r+1}$. Then C_{r+1} will be the least such $k/q_{r+1} \in \mathbb{N}$.

We take one of the two universally tight candidate $N^{k'}_{r+1}$, and as in Lemma 4.2 above we construct a universally tight R and glue in an appropriate solid torus neighborhood of a Legendrian knot L_r to obtain a universally tight N^k_r, which then glues into S^3 inductively. This shows that $N^{k'}_{r+1}$ exists.

To show that $N^{k'}_{r+1}$ fails to thicken, by Lemma 5.2 it suffices to show that $N^{k'}_{r+1}$ does not thicken to any $N^{k''}_{r+1}$, where $k'' < k'$. Inductively, we can assume N^k_r fails to thicken; in particular, the $N^{k' q_{r+1}}_r$ that contains $N^{k'}_{r+1}$ fails to thicken. Thus, if $N^{k'}_{r+1}$ admits a non-trivial thickening, it must do so inside of $N^{k' q_{r+1}}_r$. Define $M := N^{k' q_{r+1}}_r \backslash N^{k'}_{r+1}$; then M is a Seifert fibered space with one singular fiber, L_r, and with regular fibers that are topologically isotopic to the Legendrian copies of K_{r+1} on the boundary of M. M has a q_{r+1}-fold cover, $\widetilde{M}$, that is a q_{r+1}-punctured disc times S^1, where the tight contact structure admits an isotopy so that all the S^1 fibers are Legendrian with twisting $-(A_{r+1}k' + B_{r+1})$ with respect to the product framing. We can then glue in q_{r+1} standard neighborhoods of fibers with twisting $-(A_{r+1}k' + B_{r+1})$ to obtain an $S^1 \times D^2$ which itself is a standard neighborhood of a knot with twisting $-(A_{r+1}k' + B_{r+1})$. But then, if $N^{k'}_{r+1}$ thickens to a $N^{k''}_{r+1}$, where $k'' < k'$, that means that in this cover there will be a knot isotopic to one of the fibers, but with twisting greater than $-(A_{r+1}k' + B_{r+1})$, contradicting the classification of tight contact structures on solid tori. $\square$

5.2. **Case II:** $w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0$**.** As in Case I, we work through proving items 2-5 in the inductive hypothesis via a series of lemmas.

We begin by proving item 2 in the inductive hypothesis.

Lemma 5.7. *If $w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0$, then $\overline{tb}(K_{r+1}) = w(K_{r+1}) = A_{r+1}$.*

Proof. The proof is almost identical to that of step 1 in Theorem 1.5 in [L]; we will include the details, though, since certain key aspects differ. We first examine representatives of K_{r+1} at $\overline{tb}$. Since there exists a convex torus representing K_r with Legendrian divides that are (p_{r+1}, q_{r+1}) cablings (inside of the solid torus representing K_r with slope$(\Gamma) = \frac{1}{\bar{t}(K_r)}$) we know that $\overline{tb}(K_{r+1}) \geq P_{r+1}q_{r+1} = A_{r+1}$. To show that $\overline{tb}(K_{r+1}) = A_{r+1}$, we show that $\bar{t}(K_{r+1}) = 0$ by showing that the contact width $w(K_{r+1}, \mathcal{C}') = 0$, since this will yield $\overline{tb}(K_{r+1}) \leq w(K_{r+1}) = A_{r+1}$. So suppose, for contradiction, that some N_{r+1} has convex boundary with slope$(\Gamma_{\partial N_{r+1}}) = s > 0$, as measured in the $\mathcal{C}'$ framing, and two dividing curves. After shrinking N_{r+1} if necessary, we may assume that s is a large positive integer. Then let $\mathcal{A}$ be a convex annulus from ∂N_{r+1} to itself having boundary curves with slope ∞'. Taking a neighborhood of $N_{r+1} \cup \mathcal{A}$ yields a thickened torus R with boundary tori T_1 and T_2, arranged so that T_1 is inside the solid torus N_r representing K_r bounded by T_2.

Now there are no boundary parallel dividing curves on $\mathcal{A}$, for otherwise, we could pass through the bypass and increase s to ∞', yielding excessive twisting inside N_{r+1}. Hence $\mathcal{A}$ is in standard form, and consists of two parallel nonseparating arcs. We now choose a new framing $\mathcal{C}''$ for N_r where $(p_{r+1}, q_{r+1}) \mapsto (0, 1)$; then choose $(p'', q'') \mapsto (1, 0)$ so that $p''q_{r+1} - q''p_{r+1} = 1$ and such that slope$(\Gamma_{T_1}) = -s$ and slope$(\Gamma_{T_2}) = 1$. As mentioned in [EH1], this is possible since Γ_{T_1} is obtained from Γ_{T_2} by $s+1$ right-handed Dehn twists. Then note that in the $\mathcal{C}'$ framing, we have that $\frac{q_{r+1}}{p_{r+1}} >$ slope$(\Gamma_{T_2}) = \frac{q''+q_{r+1}}{p''+p_{r+1}} > \frac{q''}{p''}$, and $\frac{q_{r+1}}{p_{r+1}}$ and $\frac{q''}{p''}$ are connected by an arc in the Farey tessellation of the hyperbolic disc (see section 3.4.3 in [H]). Thus, since $\frac{1}{\bar{t}(K_r)}$ is connected by an arc to $\frac{0}{1}$ in the Farey tessellation, we must have that $\frac{q''+q_{r+1}}{p''+p_{r+1}} > \frac{1}{\bar{t}(K_r)}$. Thus we can thicken N_r to one of the solid tori with slope$(\Gamma) = -\frac{k+1}{A_rk+B_r}$ which fails to thicken. Then, just as in Claim 4.2 in [EH1], we have (i) inside R there exists a convex torus parallel to T_i with slope $\frac{q_{r+1}}{p_{r+1}}$; (ii) R can thus be decomposed into two layered *basic slices*; (iii) the tight contact structure on R must have *mixing of sign* in the Poincaré duals of the relative half-Euler classes for the layered basic slices; and (iv) this mixing of sign cannot happen inside the universally tight solid torus which fails to thicken. This last statement is due to the proof of Proposition 5.1 in [H1], where it is shown that mixing of sign will imply an overtwisted disc in the universal cover of the solid torus. Thus we have contradicted $s > 0$. So $\overline{tb}(K_{r+1}) = P_{r+1}q_{r+1} = A_{r+1}$. □

With the following lemma we prove that items 3 and 4 of the inductive hypothesis hold for K_{r+1}.

Lemma 5.8. *If $w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0$, let N_{r+1} be a solid torus representing K_{r+1}, for $r \geq 1$. Then N_{r+1} can be thickened to an $N_{r+1}^{k'}$ for some nonnegative integer k'. Moreover, if N_{r+1} fails to thicken, then it has the same boundary slope as some $N_{r+1}^{k'}$, as well as at least $2n_{r+1}^{k'}$ dividing curves.*

Proof. This is the case where $p_{r+1} < 0$ but $\frac{q_{r+1}}{p_{r+1}} \in (\frac{1}{\bar{t}(K_r)}, -\frac{1}{A_r})$; we have that $\bar{t}(K_{r+1}) = 0$. We begin as we did in case I. If N_{r+1} is a solid torus representing K_{r+1}, as before choose

L_r in $S^3\backslash N_{r+1}$ such that $\partial N(L_r)$ is joined to ∂N_{r+1} by an annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and with $tb(L_r)$ maximized over topological isotopies in the space $S^3\backslash N_{r+1}$.

Now suppose slope$(\Gamma_{\partial N(L_r)}) = -\frac{1}{m}$ where $-\frac{1}{m} < \frac{q_{r+1}}{p_{r+1}}$. Then inside $N(L_r)$ is an N_r with boundary slope $\frac{q_{r+1}}{p_{r+1}}$. But then we can extend $\mathcal{A}_{(p_{r+1},q_{r+1})}$ to an annulus that has no twisting on one edge, and we can thus thicken N_{r+1} so it has boundary slope ∞'. Moreover, since there is twisting inside $N(L_r)$, we can assure there are two dividing curves on the thickened N_{r+1}. So this situation yields no nontrivial solid tori N_{r+1} which fail to thicken.

Alternatively, suppose $-\frac{1}{m} > \frac{q_{r+1}}{p_{r+1}}$. Furthermore, for the moment suppose $-\frac{1}{m-1} > \frac{q_{r+1}}{p_{r+1}}$. Then we can use Lemma 4.4 in [H1] to conclude that there are no bypasses on the $\partial N(L_r)$-edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and so we can thicken N_{r+1} through bypasses so that $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex. Then the calculation of the boundary slope goes through as above in Lemma 5.4, and we conclude that N_{r+1} thickens to some $N^{k'}_{r+1}$. The N^k_r that is used for this will have $\frac{q_{r+1}}{p_{r+1}} < -\frac{k+1}{A_rk+B_r}$; note that such N^k_r exist since $-\frac{k+1}{A_rk+B_r} \to -\frac{1}{A_r}$ as k increases.

For the remaining case, suppose $-\frac{1}{m} > \frac{q_{r+1}}{p_{r+1}}$ and m is the least positive integer satisfying this inequality. Thus $-\frac{1}{m-1} < \frac{q_{r+1}}{p_{r+1}}$. Again look at the $\partial N(L_r)$-edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$. We claim that this edge has no bypasses. So, for contradiction, suppose it does. Then we can thicken $N(L_r)$ to a solid torus where the (efficient) geometric intersection of (p_{r+1}, q_{r+1}) with dividing curves is less than $p_{r+1} + mq_{r+1}$. Suppose the slope of this new solid torus is $-\frac{\lambda}{\mu} < -\frac{1}{m}$, where $\lambda > 1$ since m is minimized in the complement of N_{r+1}.

We do some calculations. Note first that if $\frac{m}{\mu} > 1$, then $m > \mu$, which means $m-1 \geq \mu$, which implies $-\frac{1}{m-1} \geq -\frac{1}{\mu} > -\frac{\lambda}{\mu}$, which cannot happen, again since m is minimized in the complement of N_{r+1}. Thus we must have $\frac{m}{\mu} \leq 1$. But then the geometric intersection of (p_{r+1}, q_{r+1}) with $(-\mu, \lambda)$ is $\lambda p_{r+1} + \mu q_{r+1} > \frac{\mu}{m} p_{r+1} + \mu q_{r+1} \geq \frac{m}{\mu}[\frac{\mu}{m} p_{r+1} + \mu q_{r+1}] = p_{r+1} + mq_{r+1}$. This is a contradiction.

Thus there are no bypasses on the $\partial N(L_r)$-edge of $\mathcal{A}_{(p_{r+1},q_{r+1})}$, and we can thicken N_{r+1} through any bypasses so that $\mathcal{A}_{(p_{r+1},q_{r+1})}$ is standard convex. The calculations that show N_{r+1} thickens to $N^{k'}_{r+1}$ go through as above in Lemma 5.4.

This shows that any N_{r+1} representing K_{r+1} can be thickened to one of the $N^{k'}_{r+1}$, and if N_{r+1} fails to thicken, then it has the same boundary slope as some $N^{k'}_{r+1}$. We now show that if N_{r+1} fails to thicken, and if it has the minimum number of dividing curves over all such N_{r+1} which fail to thicken and have the same boundary slope as $N^{k'}_{r+1}$, then N_{r+1} is actually an $N^{k'}_{r+1}$.

To see this, as above we can choose a Legendrian L_r that maximizes tb in the complement of N_{r+1} and such that we can join $\partial N(L_r)$ to ∂N_{r+1} by a convex annulus $\mathcal{A}_{(p_{r+1},q_{r+1})}$ whose boundaries are (p_{r+1}, q_{r+1}) and ∞' rulings on $\partial N(L_r)$ and ∂N_{r+1}, respectively. Now since N_{r+1} fails to thicken, we can assume that $\frac{q_{r+1}}{p_{r+1}} < -\frac{1}{m}$ and that there are no bypasses on the $\partial N(L_r)$-edge, and in this case we have no bypasses on the ∂N_{r+1}-edge since N_{r+1} fails to thicken and is at minimum number of dividing curves.

As above, let $N_r := N_{r+1} \cup N(\mathcal{A}_{(p_{r+1},q_{r+1})}) \cup N(L_r)$. We claim this N_r fails to thicken – the proof proceeds identically as above in Lemma 5.4, as does the proof that N_{r+1} is in fact an $N^{k'}_{r+1}$. $\square$

The following proof of item 5 of the inductive hypothesis is similar to that of Case I.

Lemma 5.9. *If $w(K_r) > \frac{P_{r+1}}{q_{r+1}} > 0$, the candidate $N^{k'}_{r+1}$ exist and actually fail to thicken for $k' \geq C_{r+1}$, where C_{r+1} is some positive integer. Moreover, these $N^{k'}_{r+1}$ have contact structures that are universally tight and have convex meridian discs whose bypasses bound half-discs all of the same sign. Also, the preferred longitude on $\partial N^{k'}_{r+1}$ has rotation number zero for $k' > 0$.*

Proof. The proof that the contact structure on a candidate $N^{k'}_{r+1}$ which fails to thicken is universally tight is identical to the argument in Case I, as is the proof that their convex meridian discs have bypasses all of the same sign, as well as the proof that the rotation number of the preferred longitude is zero.

Now we know inductively that there exists a C_r such that if $k \geq C_r$, then the N^k_r exist and fail to thicken. So suppose $k/q_{r+1} \in \mathbb{N}$ for some $k \geq C_r$. Also assume that $\frac{q_{r+1}}{p_{r+1}} < -\frac{k+1}{A_r k + B_r}$; we know such a k exists since $-\frac{k+1}{A_r k+B_r} \to -\frac{1}{A_r}$ as k increases. Then $N^{k'}_{r+1}$ exists and fails to thicken as in the argument for Case I for $k' := k/q_{r+1}$, and C_{r+1} will be the least such $k/q_{r+1} \in \mathbb{N}$. $\square$

6. Negative cablings that satisfy the UTP

We provide below the proof of Lemma 3.3, which is really just a matter of referencing a previous proof.

Proof. This is the case where $-\frac{1}{A_r} < \frac{q_{r+1}}{p_{r+1}} < 0$, we know K_r satisfies the inductive hypothesis, and we wish to show that K_{r+1} satisfies the UTP. The proof is identical to that of steps 1 and 2 in the proof of Theorem 1.5 from [L], the key being that since $-\frac{1}{A_r} < \frac{q_{r+1}}{p_{r+1}} < 0$, this cabling slope is shielded from any N^k_r that fail to thicken. $\square$

7. Transversally non-simple iterated torus knots

We have completed the UTP classification of iterated torus knots; it now remains to show that in the class of iterated torus knots, failing the UTP is a sufficient condition for supporting transversally non-simple cablings. To this end, in this section we prove Theorem 1.2; we do so by working through a series of lemmas. These lemmas will first give us information about just a piece of the Legendrian mountain range for $K_r = ((P_1, q_1), ..., (P_r, q_r))$ where $P_i > 0$ for all i; we will then use this information to obtain enough information about the Legendrian mountain ranges of certain cables K_{r+1} to conclude that these cables are transversally non-simple. We will therefore not be completing the Legendrian or transversal classification of these iterated torus knots.

Lemma 7.1. *Suppose $K_r = ((P_1, q_1), ..., (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i. Then $-\chi(K_r) = A_r - B_r$.*

Proof. A formula for $\chi(K_r)$ is given at the end of the proof of Corollary 3 in [BW]. In the notation used in that paper, the formula is $\chi(K_r) = \prod_{i=1}^r p_i - \sum_{i=1}^r q_i(p_i - 1)\prod_{j=i+1}^r p_j$, since in our case all the $e_i = 1$ as we are cabling positively at each iteration. However, note that our (P_i, q_i) corresponds to (q_i, p_i) in [BW] for $i > 1$.

Examination of this formula for $\chi(K_r)$ yields the following recursive expression using our P's and q's:

$$\chi(K_r) = q_r\chi(K_{r-1}) - P_r q_r + P_r$$

Now for a positive torus knot (P_1, q_1), we have $\chi = -A_1 + B_1$, so we can inductively assume the lemma holds for K_{r-1}. Thus using the recursive expression we have

$$\begin{aligned}
\chi(K_r) &= q_r\chi(K_{r-1}) - P_r q_r + P_r \\
&= q_r(-A_{r-1} + B_{r-1}) - A_r + q_r A_{r-1} + p_{r-1} \\
&= -A_r + B_r
\end{aligned}$$

□

In the next lemma we establish the structure of just a piece of the Legendrian mountain range for K_r:

Lemma 7.2. *Suppose $K_r = ((P_1, q_1), ..., (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i. Then there exists Legendrian representatives $L_r^{\pm}$ with $tb(L_r^{\pm}) = 0$ and $r(L_r^{\pm}) = \pm(A_r - B_r)$; also, $L_r^{\pm}$ destabilizes.*

Proof. The lemma is true for positive torus knots [EH2], so we inductively assume it is true for K_{r-1}. Then look at Legendrian rulings $\widetilde{L}_r^{\pm}$ on standard neighborhoods of the inductive $L_{r-1}^{\pm}$. In the $\mathcal{C}'$ framing the boundary slope of these $N(L_{r-1}^{\pm})$ is $-\frac{1}{A_{r-1}}$, and so a calculation shows that $t(\widetilde{L}_r^{\pm}) = -P_r$; hence $tb(\widetilde{L}_r^{\pm}) = A_r - P_r$.

To calculate the rotation number of $\widetilde{L}_r^{\pm}$, we use the following formula from [EH1], where D is a convex meridian disc for $N(L_{r-1}^{\pm})$ and Σ is a Seifert surface for the preferred longitude on $\partial N(L_{r-1}^{\pm})$:

$$\begin{aligned}
r(\widetilde{L}_r^{\pm}) &= P_r r(\partial D) + q_r r(\partial\Sigma) \\
&= \pm q_r(A_{r-1} - B_{r-1}) \\
&= \pm(q_r A_{r-1} + p_r - q_r B_{r-1} - p_r) \\
&= \pm(P_r - B_r)
\end{aligned}$$

This gives us

$$sl(T_-(\widetilde{L}_r^{+})) = (A_r - P_r) + (P_r - B_r) = A_r - B_r$$

and

$$sl(T_+(\widetilde{L}_r^{-})) = (A_r - P_r) - (-(P_r - B_r)) = A_r - B_r$$

This, along with Lemma 7.1, shows us that $\widetilde{L}_r^{+}$ is on the right-most slope of the Legendrian mountain range of K_r, and $\widetilde{L}_r^{-}$ is on the left-most edge. To the former we can perform positive stabilizations to reach L_r^{+} at $tb = 0$ and $r = A_r - B_r$; to the latter we can perform negative stabilizations to reach L_r^{-} at $tb = 0$ and $r = -(A_r - B_r)$ – we know such stabilizations can be performed since $A_r - P_r > 0$. □

So suppose K_r is an iterated torus knot that fails the UTP (which is precisely when $P_i > 0$ for all i). Then we know that for $k \geq C_r$ there exist non-thickenable solid tori N_r^k

having intersection boundary slopes of $-\frac{k+1}{A_r k+B_r}$, where these slopes are measured in the $\mathcal{C}'$ framing. Switching to the standard $\mathcal{C}$ framing, these intersection boundary slopes are $\frac{k+1}{A_r-B_r} = -\frac{k+1}{\chi(K_r)}$. Now as $k \to \infty$, there are infinitely many values of $k+1$ which are prime and greater than $A_r - B_r$. As a consequence, there are infinitely many N_r^k with two dividing curves. Based on this observation, we make the following definition:

Definition 7.3. Suppose $K_r = ((P_1, q_1), ..., (P_r, q_r))$ is an iterated torus knot where $P_i > 0$ for all i. Let $\widehat{K}_{r+1}$ be a cabling of K_r with $\mathcal{C}'$ slope $-\frac{k+1}{A_r k+B_r}$, where $-\frac{1}{A_r-1} < -\frac{k+1}{A_r k+B_r} < -\frac{1}{A_r}$ and there is an N_r^k with two dividing curves that fails to thicken.

So given K_r, there are infinitely many such cabling knot types $\widehat{K}_{r+1}$, all of these being cablings of the form $(-\chi(K_r), k+1)$ as measured in the preferred framing. The following lemma will then prove Theorem 1.2.

Lemma 7.4. *$\widehat{K}_{r+1}$ is a transversally non-simple knot type.*

Proof. We first calculate $\chi(\widehat{K}_{r+1})$. Using the recursive expression we obtain

$$\begin{aligned}
\chi(\widehat{K}_{r+1}) &= q_{r+1}\chi(K_r) - P_{r+1}q_{r+1} + P_{r+1} \\
&= (k+1)(-A_r + B_r) - (A_r - B_r)(k+1) + (A_r - B_r) \\
&= (2k+1)(-A_r + B_r)
\end{aligned}$$

We now look at the two universally tight non-thickenable N_r^k that have representatives of $\widehat{K}_{r+1}$ as Legendrian divides. These Legendrian divides have $tb = A_{r+1} = q_{r+1}P_{r+1} = (k+1)(A_r - B_r)$. To calculate rotation numbers, we have two possibilities, depending on which boundary of the two universally tight N_r^k the Legendrian divides reside. Using the formula from [EH1], we obtain

$$\begin{aligned}
r(\widehat{K}_{r+1}) &= q_{r+1}r(\partial\Sigma) + P_{r+1}r(\partial D) \\
&= P_{r+1}(\pm(q_{r+1} - 1)) \\
&= \pm k(A_r - B_r)
\end{aligned}$$

We will call the two Legendrian divides corresponding to $r = \pm k(A_r - B_r)$, $L_{r+1}^{\pm}$ respectively. We can calculate the self-linking number for the negative transverse push-off of L_{r+1}^{+} to be $sl = (2k+1)(A_r - B_r) = -\chi(\widehat{K}_{r+1})$. This shows that L_{r+1}^{+} is on the right-most edge of the Legendrian mountain range and is at $\overline{tb}$. Similarly, L_{r+1}^{-} is on the left-most edge of the Legendrian mountain range and is at $\overline{tb}$.

We now look at solid tori $\widehat{N}_r$ with intersection boundary slope $-\frac{k+1}{A_r k+B_r}$, but which *thicken* to solid tori with intersection boundary slopes $-\frac{1}{A_r-1}$. Such tori $\partial\widehat{N}_r$ are embedded in universally tight basic slices bounded by tori with dividing curves of slope $-\frac{1}{A_r-1}$ and $-\frac{1}{A_r}$. Legendrian divides on such $\widehat{N}_r$ have $tb = (k+1)(A_r - B_r)$; to calculate possible rotation numbers for these Legendrian divides, we recall the procedure used in the proof of Theorem 1.5 in [L]. There we used a formula for the rotation numbers from [EH1], where the range of rotation numbers was given by the following (substituting $A_r - 1$ for n):

$$r(L_{r+1}) \in \{\pm(p_{r+1} + (A_r - 1)q_{r+1} + q_{r+1}r(L_r)) | tb(L_r) = A_r - (A_r - 1) = 1\}$$

Now from Lemma 7.2 we know that there is an L_r with $tb(L_r) = 1$ and $r(L_r) = -(A_r - B_r) + 1$. Plugging this value of the rotation number into the expression above yields $r(L_{r+1}) = \pm k(A_r - B_r)$. We will call the Legendrian divides having these rotation numbers $\widehat{L}^{\pm}_{r+1}$, respectively. Important for our purposes is that $\widehat{L}^{\pm}_{r+1}$ have the same values of tb and r as $L^{\pm}_{r+1}$.

We focus in, for the sake of argument, on L^-_{r+1} and $\widehat{L}^-_{r+1}$, and we show that $T_-(L^-_{r+1})$ is not transversally isotopic to $T_-(\widehat{L}^-_{r+1})$, despite having the same self-linking number.

Consider first $T_+(L^-_{r+1})$. It is in fact one of the dividing curves on ∂N^k_r, and is also at maximal self-linking number for $\widehat{K}_{r+1}$. Similarly, $T_+(\widehat{L}^-_{r+1})$ is one of the dividing curves on $\partial \widehat{N}_r$, and is also at maximal self-linking number. Now from [He1] we know that $\widehat{K}_{r+1}$ is a fibered knot that supports the standard contact structure, since it is an iterated torus knot obtained by cabling positively at each iteration. As a consequence, from [EV], we also know that $\widehat{K}_{r+1}$ has a unique transversal isotopy class at $\overline{sl}$. Hence we know that $T_+(L^-_{r+1})$ and $T_+(\widehat{L}^-_{r+1})$ are transversally isotopic. Thus there is a transverse isotopy (inducing an ambient contact isotopy) that takes these two dividing curves on the two different tori to each other. Thus we may assume that ∂N^k_r and $\partial \widehat{N}_r$ intersect along one component of the dividing curves; we call this component γ_+.

Now suppose, for contradiction, that $T_-(L^-_{r+1})$ is transversally isotopic to $T_-(\widehat{L}^-_{r+1})$. These transverse knots are represented by the other two non-intersecting dividing curves on ∂N^k_r and $\partial \widehat{N}_r$, respectively, and there is a transverse isotopy taking one to the other. We claim that this transverse isotopy can be performed relative to γ_+. To see this, note that associated to $S^3 \backslash N(\gamma_+)$ is an open book decomposition of S^3, with pages being Seifert surfaces Σ for the knot γ_+. Moreover, the standard contact structure is supported by this open book decomposition. Thus the transverse isotopy taking $T_-(L^-_{r+1})$ to $T_-(\widehat{L}^-_{r+1})$ will induce an ambient isotopy of open book decompositions supporting the standard contact structure, all with a transversal representative of γ_+ on the binding. Since ∂N^k_r is incompressible in $S^3 \backslash N(\gamma_+)$, it is therefore evident that the isotopy taking $T_-(L^-_{r+1})$ to $T_-(\widehat{L}^-_{r+1})$ can be accomplished simply as an isotopy of ∂N^k_r relative to γ_+.

Thus we may assume that after a contact isotopy of S^3, ∂N^k_r and $\partial \widehat{N}_r$ intersect along their two dividing curves, which we denote as γ_+ and γ_-, and we observe that there is an isotopy (not necessarily a contact isotopy) of N^k_r to $\widehat{N}_r$ relative to γ_+ and γ_-. We claim that as a result $\widehat{N}_r$ cannot thicken, thus obtaining our contradiction. We do this by noting that the isotopy of N^k_r to $\widehat{N}_r$ relative to γ_+ and γ_- may be accomplished by the attachment of successive bypasses. Since these bypasses are attached in the complement of the two dividing curves, none of these bypass attachments can change the boundary slope. However, they may increase or decrease the number of dividing curves. Starting with $T = \partial N^k_r$, we make the following inductive hypothesis, which we will prove is maintained after bypass attachments:

1. T is a convex torus which contains γ_+ and γ_-, and thus has slope $-\frac{k+1}{A_r k + B_r}$.
2. T is a boundary-parallel torus in a $[0, 1]$-invariant $T^2 \times [0, 1]$ with $\text{slope}(\Gamma_{T_0}) = \text{slope}(\Gamma_{T_1}) = -\frac{k+1}{A_r k + B_r}$, where the boundary tori have two dividing curves.
3. There is a contact diffeomorphism $\phi : S^3 \to S^3$ which takes $T^2 \times [0, 1]$ to a standard I-invariant neighborhood of ∂N^k_r and matches up their complements.

The argument that follows is similar to Lemma 6.8 in [EH1]. First note that item 1 is preserved after a bypass attachment, since such a bypass is in the complement of γ_+ and γ_-, and thus cannot change the slope of the dividing curves. To see that items 2 and 3 are preserved, suppose that T' is obtained from T by a single bypass. Since the slope was not changed, such a (non-trivial) bypass must either increase or decrease the number of dividing curves by 2. Suppose first that the bypass is attached from the inside, so that $T' \subset N$, where N is the solid torus bounded by T. For convenience, suppose $T = T_{0.5}$ inside the $T^2 \times [0,1]$ satisfying items 2 and 3 of the inductive hypothesis. Then we form the new $T^2 \times [0.5,1]$ by taking the old $T^2 \times [0.5,1]$ and adjoining the thickened torus between T and T'. Now T' bounds a solid torus N', and, by the classification of tight contact structures on solid tori, we can factor a nonrotative outer layer which is the new $T^2 \times [0, 0.5]$.

Alternatively, if $T' \subset (S^3 \backslash N)$, then we know that N' thickens to an N_r^k, and thus there exists a nonrotative outer layer $T^2 \times [0.5, 1]$ for $S^3 \backslash N'$, where T_1 has two dividing curves. Thus the proof is done, for after enough bypass attachments we will obtain $T = \partial \widehat{N}_r$, with $\widehat{N}_r$ non-thickenable. But this is a contradiction, since $\widehat{N}_r$ does thicken. $\square$

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References

[BW] J. Birman and N. Wrinkle, *On transversally simple knots*, Journal of Differential Geometry 55 (2000), 325-354.

[EFM] J. Epstein, D. Fuchs, and M. Meyer, *Chekanov-Eliashberg invariants and transverse approximations of Legendrian knots*, Pac. J. Math. 201 (1) (2001), 89-106.

[E] J. Etnyre, *Lectures on open book decompositions and contact structures*, e-print at arXiv:math/0409402v3.

[EH1] J. Etnyre and K. Honda, *Cabling and transverse simplicity*, Ann. of Math. (2) 162 (2005), no. 3, 1305–1333.

[EH2] J. Etnyre and K. Honda, *Knots and Contact Geometry I: Torus Knots and the Figure Eight Knot*, Journal of Symplectic Geometry, Vol. 1, No. 1 (2001), 63-120.

[EV] J. Etnyre and J. Van Horn-Morris, *Fibered transverse knots and the Bennequin bound*, (2008), e-print at arXiv:0803.0758v2.

[G] E. Giroux, *Convexité en topologie de contact*, Comm. Math. Helv. 66 (1991), 615-689.

[He1] M. Hedden, *Some remarks on cabling, contact structures, and complex curves*, Proc. of 13th Gokova Geometry-Topology Conference (2008), 1-11.

[He2] M. Hedden, *An Ozsváth-Szabó Floer homology invariant of knots in contact manifolds*, e-print at arXiv:0708.0448.

[He3] M. Hedden, *Notions of positivity and the Ozsváth-Szabó concordance invariant*, e-print at arxiv:math/0509499.

[H1] K. Honda, *On the classification of tight contact structures I*, Geometry & Topology 4 (2000), 309-368.

[H2] K. Honda, *On the classification of tight contact structures II*, Journal of Differential Geometry 55 (2000), 83-143.

[L] D. LaFountain, *Studying uniform thickness I: Legendrian simple iterated torus knots*, e-print at arXiv:0905.2760.

[Mi] J. Milnor, *Singular points of complex surfaces*, Princeton Univ. Press, Princeton 1968.

[TW] W.P. Thurston and H.E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52, (1975), 345-347.